Chapter 2

Lecture 1

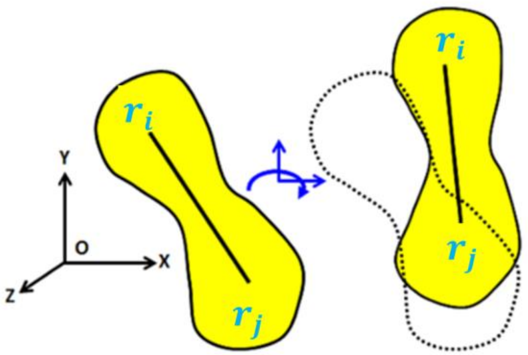
Lagrange's Mechanics

Dr. Akhlaq Hussain

2.1 Constraints

Constraints are the geometrical or kinematical restrictions on the motion of the particle **OR** system of the particles.

Such system is called **Constrained systems** and their motion is known as **constrained or restricted motion**.



$|r_i - r_j|^2 - C_{ij}^2 = 0$

Rigid body → distance between any two particles remains unchanged.
Gas molecules → within the container is restricted by the walls of the vessels.

Classification of Constraints

Holonomic constraints:- Constraints are said to be holonomic if the conditions of all the constraints can be expressed as equations connecting the coordinates of the particles and possible time in the form

$$f\,(r_1, r_2, r_3 \ldots \ldots .., r_n, t) = 0 \qquad (2.1)$$

In Cartesian coordinates equation (2.1) can be written as,

$$f\,(x_1, y_1, z_1;\ x_2, y_2, z_2, \ldots \ldots \ldots\ x_n, y_n, z_n, t) = 0 \qquad (2.2)$$

2.1 Constraints

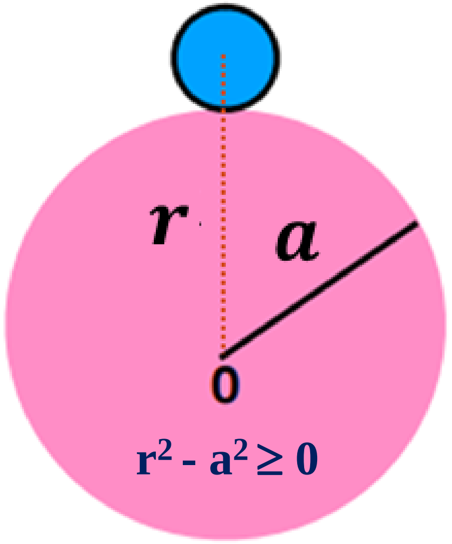
Non-holonomic constraints: - If the conditions of the constraints can not be expressed as equations connecting the coordinates of particles as in case of holonomic, they are called as non-holonomic constraints.

The conditions of these constraints are expressed in the form of inequalities.

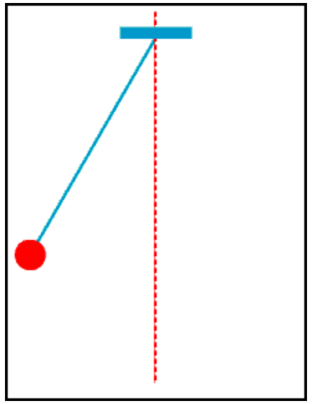
$$f(r_1, r_2, r_3 \ldots\ldots\ldots, r_n, t) \neq 0 \qquad \textbf{(2.3)}$$

Examples of non-holonomic constraints

1. Constraints involved in the motion of a particle placed on the surface of a solid sphere $r^2 - a^2 \geq 0$. **(2.4)**
2. An object rolling on the rough surface without slipping.
3. Constraints involved in the motion of gas molecules in a container.



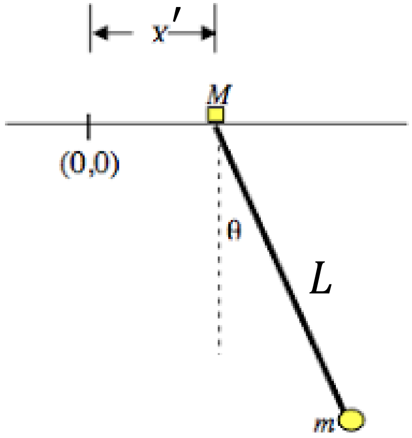
Constraints



Scleronomic and Rheonomic Constraints: - The constraints which are independent of time are called Scleronomic constraints and the constraints which contain time explicitly, called rheonomic constraints

Examples: - A bead sliding on a rigid curved wire fixed in space is obviously subjected to Scleronomic constraints and if the wire is moving is prescribed fashion the constraints become Rheonomic.

Consider a pendulum of constant length and fixe pivot point



$$x^2 + y^2 - L^2 = 0$$ Scleronomic Constraint

If the pivot point is moving along x axis

$$x' = x_o \cos wt$$

The equation of motion will be

$$(x - x_o \cos wt)^2 + y^2 - L^2 = 0$$ Rheonomic Constraint

Problems Due to Constraints

1. The coordinates are no longer independent.

   They are connected by equation of constraints for rigid body

   $$r_i - r_j \,|^2 - C_{ij}^2 = 0$$

   $$x^2 + y^2 - L^2 = 0$$

2. To apply Newton's 2^{nd} Law, we need total force acting on each particle. Force of Constraints are not known or easily calculated.

   Forces are vector quantities; the vector nature of forces also makes it difficult to solve the problem

Generalized Coordinates

How to solve problems associated with Constraints…..

Consider a system of N-particles. If each particle has 3-degrees of freedom.

Total freedom of system is *3N*

If k is the number of holonomic constraints on the system.

The total number of independent coordinates $s = 3N - k$

We define "s" number of independent coordinates

$q_1, q_2, q_3, \dots \dots\ q_{s},$ or just $q_i\ where\ i = 1,2,3,4 \dots s$)

Such that

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q_1, q_2, q_3, \dots \dots q_s)$$

$$q_i = q_i(r_1, r_2, r_3, \dots \dots r_n)$$

$$x = rsin\,\theta\,cos\varphi$$

$$x = x(r, \theta, \varphi)$$

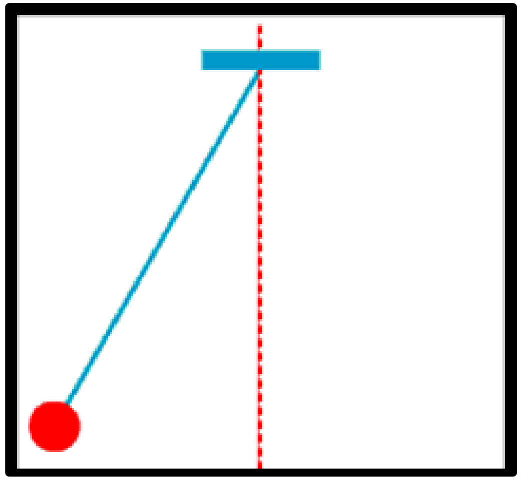
$$y = rsin\,\theta \sin\varphi$$

$$y = y(r, \theta, \varphi)$$

$$z = rcos\,\theta$$

$$z = z(r, \theta)$$

Generalized Coordinates



In case of pendulum

$r = L = constant$

Independent coordinate is $q_1 = \theta$

For spherical pendulum with constant length

The coordinates (r, θ, ϕ) since r = constant

We have

$(q_1, q_2) = (\theta, \phi)$ are independent coordinates,

To overcome the second difficulty, we should formulate the mechanics such that the unknow constraint forces disappeared in calculation.

We will be using energy (K.E +P.E), Momentum and position to solve that system.

Generalized Coordinates

Suppose a system of N-particles. If system has n-degree (n=3N-k) of freedom then we need n-generalized coordinates $q_1, q_2, q_3, \ldots \ldots q_n$ to specify the configuration of holonomic dynamical system. They may be cartesian or spherical polar coordinates etc.

The configuration of the system is expressed as function of the generalized coordinates.

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q_1, q_2, q_3, \ldots \ldots q_n, t)$$

If the system moves from one configuration $(q_1, q_2, q_3, \ldots \ldots q_n, t)$ to a neighboring configuration $(q_1 + \delta q_1, q_2 + \delta q_2, q_3 + \delta q_3, \ldots \ldots q_n + \delta q_n)$

$$\boldsymbol{r}_i + \boldsymbol{\delta r}_i = \boldsymbol{r}_i(q_1 + \delta q_1, q_2 + \delta q_2, q_3 + \delta q_3, \ldots \ldots q_n + \delta q_n) \text{ where } \delta t = 0$$

$$\boldsymbol{\delta r}_i = \frac{\partial \boldsymbol{r_i}}{\partial q_1}\delta q_1 + \frac{\partial \boldsymbol{r_i}}{\partial q_2}\delta q_2 + \cdots\cdots\cdots\cdots + \frac{\partial \boldsymbol{r_i}}{\partial q_n}\delta q_n$$

$$\boxed{\boldsymbol{\delta r}_i = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r_i}}{\partial q_j}\delta q_j} \text{ (Virtual Displacement)}$$

Virtual Work

virtual Displacement A virtual displacement is an arbitrary instantons, infinitesimal displacement of a dynamical system. Independent of time and consistent with the constraints of the system.

$$\boldsymbol{\delta r}_i = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r_i}}{\partial q_j} \delta q_j$$

Principle of virtual work

A system under workless constraints is in equilibrium under applied forces, if and only if zero virtual work is done by the applied forces in an arbitrary infinitesimal displacement satisfying constraints.

$$\sum_i \boldsymbol{F_i} \cdot \delta \boldsymbol{r}_i = 0$$

where

$$\boldsymbol{F_i} = \boldsymbol{F_i}^{(\boldsymbol{e})} + \sum_j \boldsymbol{F_{ji}}$$

Generalized Velocity

The time derivative of the generalized coordinates is called generalized velocity associated with co-ordinates for an unconstrained system, For a system with n-degree of freedom and defined with configuration

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q_1, q_2, q_3, \ldots \ldots q_n, t)$$

The velocity is

$$\frac{d\boldsymbol{r}_i}{dt} = \frac{\partial \boldsymbol{r}_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial \boldsymbol{r}_i}{\partial q_2}\frac{dq_2}{dt} + \cdots + \frac{\partial \boldsymbol{r}_i}{\partial q_n}\frac{dq_n}{dt} + \frac{\partial \boldsymbol{r}_i}{\partial t}\frac{dt}{dt}$$

$$\dot{\boldsymbol{r}}_i = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \boldsymbol{r}_i}{\partial t}$$

Where $\dot{q}_j$ is generalized velocity.

Generalized Acceleration

Components of generalized acceleration are obtained by differentiating above equation

$$\ddot{\boldsymbol{r}}_i = \frac{d\dot{\boldsymbol{r}}_i}{dt} = \frac{d}{dt}\left(\sum_{j=1}^{n}\frac{\partial \boldsymbol{r_i}}{\partial q_j}\dot{q}_j + \frac{\partial \boldsymbol{r_i}}{\partial t}\right) = \sum_{j=1}^{n}\frac{\partial \boldsymbol{r_i}}{\partial q_j}\frac{d}{dt}\dot{q}_j + \sum_{j=1}^{n}\frac{d}{dt}\frac{\partial \boldsymbol{r_i}}{\partial q_j}\dot{q}_j + \frac{d}{dt}\frac{\partial \boldsymbol{r_i}}{\partial t}$$

$$\ddot{\boldsymbol{r}}_i = \sum_{j=1}^{n}\frac{\partial \boldsymbol{r_i}}{\partial q_j}\ddot{q}_j + \sum_{j=1}^{n}\frac{\partial}{\partial q_j}\frac{d\boldsymbol{r_i}}{dt}\dot{q}_j + \frac{\partial}{\partial t}\frac{d\boldsymbol{r_i}}{dt}$$

$$\ddot{\boldsymbol{r}}_i = \sum_{j=1}^{n}\frac{\partial \boldsymbol{r_i}}{\partial q_j}\ddot{q}_j + \sum_{j=1}^{n}\frac{\partial}{\partial q_j}\left(\frac{\partial \boldsymbol{r_i}}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial \boldsymbol{r_i}}{\partial q_2}\frac{dq_2}{dt} + \cdots + \frac{\partial \boldsymbol{r_i}}{\partial q_n}\frac{dq_n}{dt} + \frac{\partial \boldsymbol{r_i}}{\partial t}\right)\dot{q}_j$$
$$+ \frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{r_i}}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial \boldsymbol{r_i}}{\partial q_2}\frac{dq_2}{dt} + \cdots + \frac{\partial \boldsymbol{r_i}}{\partial q_n}\frac{dq_n}{dt} + \frac{\partial \boldsymbol{r_i}}{\partial t}\right)$$

$$\ddot{\boldsymbol{r}}_i = \sum_{j=1}^{n}\frac{\partial \boldsymbol{r_i}}{\partial q_j}\ddot{q}_j + \sum_{j=1}^{n}\frac{\partial}{\partial q_j}\left(\sum_{k=1}^{n}\frac{\partial \boldsymbol{r_i}}{\partial q_k}\dot{q}_k + \frac{\partial \boldsymbol{r_i}}{\partial t}\right)\dot{q}_j + \frac{\partial}{\partial t}\left(\sum_{k=1}^{n}\frac{\partial \boldsymbol{r_i}}{\partial q_k}\dot{q}_k + \frac{\partial \boldsymbol{r_i}}{\partial t}\right)$$

$$\ddot{\boldsymbol{r}}_i = \sum_{j=1}^{n}\frac{\partial \boldsymbol{r_i}}{\partial q_j}\ddot{q}_j + \sum_{j,k=1}^{n}\frac{\partial^2 \boldsymbol{r_i}}{\partial q_k \partial q_j}\dot{q}_k\dot{q}_j + \sum_{j=1}^{n}\frac{\partial^2 \boldsymbol{r_i}}{\partial t\partial q_j}\dot{q}_j + \sum_{k=1}^{n}\frac{\partial^2 \boldsymbol{r_i}}{\partial t\partial q_k}\dot{q}_k + \frac{\partial^2 \boldsymbol{r_i}}{\partial t^2}$$

$$\ddot{\boldsymbol{r}}_i = \sum_{j=1}^{n}\frac{\partial \boldsymbol{r_i}}{\partial q_j}\ddot{q}_j + \sum_{j,k=1}^{n}\frac{\partial^2 \boldsymbol{r_i}}{\partial q_k \partial q_j}\dot{q}_k\dot{q}_j + 2\sum_{j=1}^{n}\frac{\partial^2 \boldsymbol{r_i}}{\partial t\partial q_j}\dot{q}_j + \frac{\partial^2 \boldsymbol{r_i}}{\partial t^2}$$

Above equation makes it clear that the cartesian components are not linear functions of components of generalized acceleration $\ddot{q}_i$ alone, but depend quadratically and linearly on generalized velocity component as $\dot{q}_i$well

D' Alembert Principle

The principle state that the particle will be in equilibrium under a force $\boldsymbol{F_i}$ *equal to the actual force plus a reverse effective force* $\dot{\boldsymbol{p}}_{\boldsymbol{i}}$

$$\boldsymbol{F_i} = \dot{\boldsymbol{p}}_i$$

$$\boldsymbol{F_i} - \dot{\boldsymbol{p}}_i = \boldsymbol{0}$$

$$\sum_i (\boldsymbol{F_i} - \dot{\boldsymbol{p}}_i) \cdot \delta \boldsymbol{r}_i = 0 \qquad \text{where} \qquad \boldsymbol{F_i} = \boldsymbol{F_i}^{(e)} + \sum_j \boldsymbol{F_{ji}}$$

$$\sum_i \boldsymbol{F_i}^{(e)} \cdot \delta \boldsymbol{r}_i + \sum_{i,j} \boldsymbol{F_{ji}} \cdot \delta \boldsymbol{r}_i - \sum_i \dot{\boldsymbol{p}}_{\boldsymbol{i}} \cdot \delta \boldsymbol{r}_i = 0$$

If we restrict ourselves to workless constraints.

$$\sum_i \boldsymbol{F_i}^{(e)} \cdot \delta \boldsymbol{r}_i + \underbrace{\sum_{i,j} \boldsymbol{F_{ji}} \cdot \delta \boldsymbol{r}_i}_{\boldsymbol{0}} - \sum_i \dot{\boldsymbol{p}}_{\boldsymbol{i}} \cdot \delta \boldsymbol{r}_i = 0$$

$$\boxed{\sum_i \left(\boldsymbol{F_i}^{(e)} - \dot{\boldsymbol{p}}_i\right) \cdot \delta \boldsymbol{r}_i = 0}$$

Lagrange's Equation

Suppose a system of N-particles having masses $m_1, m_2, m_3, \ldots \ldots m_N$ at position $r_1, r_2, r_3, \ldots \ldots r_N$ respectively. If system has n-degree (n=3N-k) of freedom then we need n-generalized coordinates $q_1, q_2, q_3, \ldots \ldots q_n$ to specify the configuration of holonomic dynamical system.

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q_1, q_2, q_3, \ldots \ldots q_n, t)$$

$$\frac{\boldsymbol{dr}_i}{\boldsymbol{dt}} = \dot{\boldsymbol{r}}_i = \frac{\partial \boldsymbol{r_i}}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial \boldsymbol{r_i}}{\partial q_2}\frac{dq_2}{dt} + \cdots + \frac{\partial \boldsymbol{r_i}}{\partial q_n}\frac{dq_n}{dt} + \frac{\partial \boldsymbol{r_i}}{\partial t} = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r_i}}{\partial q_j}\dot{q}_j + \frac{\partial \boldsymbol{r_i}}{\partial t}$$

$$\frac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial \boldsymbol{r_i}}{\partial q_1}\dot{q}_1 + \frac{\partial \boldsymbol{r_i}}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial \boldsymbol{r_i}}{\partial q_j}\dot{q}_j + \cdots + \frac{\partial \boldsymbol{r_i}}{\partial q_n}\dot{q}_n + \frac{\partial \boldsymbol{r_i}}{\partial t}\right) = \frac{\partial \boldsymbol{r_i}}{\partial q_j}\frac{\partial \dot{q}_j}{\partial \dot{q}_j} = \frac{\partial \boldsymbol{r_i}}{\partial q_j}$$

$$\boxed{\frac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_j} = \frac{\partial \boldsymbol{r_i}}{\partial q_j}}$$

Lagrange's Equation

Considering the virtual displacement $=\boldsymbol{\delta r}_i = \frac{\partial \boldsymbol{r_i}}{\partial q_1}\delta q_1 + \frac{\partial \boldsymbol{r_i}}{\partial q_2}\delta q_2 + \cdots \cdots + \frac{\partial \boldsymbol{r_i}}{\partial q_n}\delta q_n$

$$\boldsymbol{\delta r}_i = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r_i}}{\partial q_j}\delta q_j$$

Let the virtual work done by force $\boldsymbol{F}_i = m_i \ddot{\boldsymbol{r}}_i$

$$\sum_i \boldsymbol{F}_i . \boldsymbol{\delta r}_i = \sum_{i,j} \boldsymbol{F}_i . \frac{\partial \boldsymbol{r_i}}{\partial q_j}\delta q_j$$

$$\sum_{i=1}^{N} \boldsymbol{F}_i . \boldsymbol{\delta r}_i = \sum_{j=1}^{n} Q_j \delta q_j \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 1$$

$$Q_j = \sum_{i=1}^{N} \boldsymbol{F}_i . \frac{\partial \boldsymbol{r_i}}{\partial q_j}$$

Q_j is generalized force whose dimensions are not necessarily equal to the force. It may be force or torque. Now using the second term of D' Alembert principle

$$\boxed{\sum_{i=1}^{N} \dot{\boldsymbol{P}}_i . \boldsymbol{\delta r}_i = \sum_{i,j} m_i \ddot{\boldsymbol{r}}_i . \frac{\partial \boldsymbol{r_i}}{\partial q_j}\delta q_j}$$

Lagrange's Equation

Considering the differential equation

$$\sum_i^N \frac{d}{dt}\left[m_i\dot{\boldsymbol{r}}_i.\frac{\partial \boldsymbol{r_i}}{\partial q_j}\right] = \sum_i^N m_i\ddot{\boldsymbol{r}}_i.\frac{\partial \boldsymbol{r_i}}{\partial q_j} + \sum_i^N m_i\dot{\boldsymbol{r}}_i.\frac{d}{dt}\frac{\partial \boldsymbol{r_i}}{\partial q_j}$$

$$\sum_i^N m_i\ddot{\boldsymbol{r}}_i.\frac{\partial \boldsymbol{r_i}}{\partial q_j} = \sum_i^N \frac{d}{dt}\left[m_i\dot{\boldsymbol{r}}_i.\frac{\partial \boldsymbol{r_i}}{\partial q_j}\right] - \sum_i^N m_i\dot{\boldsymbol{r}}_i.\frac{\partial \dot{\boldsymbol{r}}_{\boldsymbol{i}}}{\partial q_j}$$

$$\boxed{\frac{\partial \dot{\boldsymbol{r}}_{\boldsymbol{i}}}{\partial \dot{q}_j} = \frac{\partial \boldsymbol{r_i}}{\partial q_j}}$$

$$\sum_i^N m_i\ddot{\boldsymbol{r}}_i.\frac{\partial \boldsymbol{r_i}}{\partial q_j} = \sum_i^N \frac{d}{dt}\left[m_i\dot{\boldsymbol{r}}_i.\frac{\partial \dot{\boldsymbol{r}}_{\boldsymbol{i}}}{\partial \dot{q}_j}\right] - \sum_i^N m_i\dot{\boldsymbol{r}}_i.\frac{\partial \dot{\boldsymbol{r}}_{\boldsymbol{i}}}{\partial q_j}$$

$$\sum_i^N m_i\ddot{\boldsymbol{r}}_i.\frac{\partial \boldsymbol{r_i}}{\partial q_j} = \sum_i^N \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}\left[\frac{1}{2}m_i\dot{\boldsymbol{r}}_i{}^2\right] - \sum_i^N \frac{\partial}{\partial q_j}\left[\frac{1}{2}m_i\dot{\boldsymbol{r}}_i{}^2\right]$$

$$\sum_i^N m_i\ddot{\boldsymbol{r}}_i.\frac{\partial \boldsymbol{r_i}}{\partial q_j} = \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}\left[\sum_i^N \frac{1}{2}m_i\dot{\boldsymbol{r}}_i{}^2\right] - \frac{\partial}{\partial q_j}\left[\sum_i^N \frac{1}{2}m_i\dot{\boldsymbol{r}}_i{}^2\right]$$

$$\sum_i^N m_i\ddot{\boldsymbol{r}}_i.\frac{\partial \boldsymbol{r_i}}{\partial q_j} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}$$

$$\sum_{i=1}^N \dot{\boldsymbol{P}}_i.\boldsymbol{\delta r}_i = \sum_{i,j} m_i\ddot{\boldsymbol{r}}_i.\frac{\partial \boldsymbol{r_i}}{\partial q_j}\delta q_j = \sum_{j=1}^n\left[\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}\right]\delta q_j \ldots\ldots\ldots\ldots.2$$

Lagrange's Equation

Using eq 1 and eq 2 in the D' Alembert principle

$$\sum_i \left(\boldsymbol{F_i}^{(e)} - \dot{\boldsymbol{p}}_i\right) \cdot \delta \boldsymbol{r}_i = \sum_{j=1}^{n} Q_j \delta q_j - \sum_{j=1}^{n} \left[\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}\right] \delta q_j = 0$$

$$\sum_i \left(\boldsymbol{F_i}^{(e)} - \dot{\boldsymbol{p}}_i\right) \cdot \delta \boldsymbol{r}_i = \sum_{j=1}^{n} \left[Q_j - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} + \frac{\partial T}{\partial q_j}\right] \delta q_j = 0$$

$$\boxed{Q_j = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}}$$

Above equation is known as Lagrange's equation.

Where Q_j is generalized force. Its either

i) Gravitational Force
ii) Spring force
iii) External applied
iv) Electric or magnetic force
v) Torque

Lagrange’s Mechanics Examples

A Particle of mass “*m*” moves in a plane. Find its equation in cartesian coordinates.

Solution: Consider the coordinates of particle having mass m is r(x,y) *or* $\bar{r} = x\hat{\imath} + y\hat{\jmath}$ in plane. Let the force acting in x and y direction be F_x and F_y respectively.

Kinetic energy = $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

Now Lagrange’s Equation $\quad Q_j = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}$ can be written as

For x coordinate $\quad Q_x = \frac{d}{dt}\frac{\partial T}{\partial \dot{x}} - \frac{\partial T}{\partial x} \quad$ & $\quad Q_x = F_x \frac{\partial \bar{r}}{\partial x} = F_x\hat{\imath}$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} - \frac{\partial T}{\partial x} = \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\left[\frac{1}{2}m(\dot{x}^2 + \dot{y}^2)\right] - \overbrace{\frac{\partial}{\partial x}\left[\frac{1}{2}m(\dot{x}^2 + \dot{y}^2)\right]}^{0}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} - \frac{\partial T}{\partial x} = \frac{d}{dt}(m\dot{x}) = m\ddot{x}$$

Therefore, $F_x\hat{\imath} = m\ddot{x}$

Lagrange’s Mechanics Examples

For y coordinate

$$Q_y = \frac{d}{dt}\frac{\partial T}{\partial \dot{y}} - \frac{\partial T}{\partial y} \qquad \& \qquad Q_y = F_y \frac{\partial \bar{r}}{\partial y} = F_y \hat{j}$$

0

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{y}} - \frac{\partial T}{\partial y} = \frac{d}{dt}\frac{\partial}{\partial \dot{y}}\left[\frac{1}{2}m(\dot{x}^2 + \dot{y}^2)\right] - \frac{\partial}{\partial y}\left[\frac{1}{2}m(\dot{x}^2 + \dot{y}^2)\right]$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{y}} - \frac{\partial T}{\partial y} = \frac{d}{dt}(m\dot{y}) = m\ddot{y}$$

Or $\boxed{F_y \hat{j} = m\bar{\ddot{y}}}$

Chapter 2

Lecture 2

Lagrange's Mechanics

Dr. Akhlaq Hussain

Lagrange's Mechanics Examples

A Particle of mass "*m*" moves in a plane. Find its equation in plane polar coordinates.

Solution: Consider the coordinates of particle having mass "*m*" are (r,θ) in plane. Let the force acting in "r" and "θ" direction be "F_r" and "F_θ" respectively.

Kinetic energy = $T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$

Now Lagrange's Equation $Q_j = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}$ can be written as

For r coordinate $Q_r = \frac{d}{dt}\frac{\partial T}{\partial \dot{r}} - \frac{\partial T}{\partial r}$ & $Q_r = F_r\frac{\partial \bar{r}}{\partial r} = F_r\hat{r}$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{r}} - \frac{\partial T}{\partial r} = \frac{d}{dt}\frac{\partial}{\partial \dot{r}}\left[\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)\right] - \frac{\partial}{\partial r}\left[\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)\right]$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{r}} - \frac{\partial T}{\partial r} = \frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 = m\left(\ddot{r} - r\dot{\theta}^2\right)$$

Therefore, $F_r\hat{r} = m\left(\ddot{r} - r\dot{\theta}^2\right)\hat{r}$

Lagrange's Mechanics Examples

For θ coordinate

$$Q_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} \qquad \& \qquad Q_\theta = F_\theta \frac{\partial \bar{r}}{\partial \theta} = F_\theta r \frac{\partial \hat{r}}{\partial \theta} = F_\theta r \hat{\theta}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} = \frac{d}{dt}\frac{\partial}{\partial \dot{\theta}}\left[\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)\right] - \frac{\partial}{\partial \theta}\left[\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)\right]$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} = \frac{d}{dt}\left(mr^2\dot{\theta}\right) = m\left(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}\right)$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} = mr\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$$

Or $$F_\theta r = mr\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$$

$$F_\theta \hat{\theta} = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta}$$

Lagrange's Equation For conservative force

For conservative force which can be derivable from a scalar Potential

$$\vec{\boldsymbol{F}}_i = -\vec{\nabla}_i V_i$$

$$\Rightarrow Q_j = \sum_{i=1}^{N} \boldsymbol{F}_i . \frac{\partial \boldsymbol{r_i}}{\partial q_j} = -\sum_{i=1}^{N} \vec{\nabla}_i V_i . \frac{\partial \boldsymbol{r_i}}{\partial q_j}$$

$$\Rightarrow Q_j = -\sum_{i=1}^{N} \frac{\partial V_i}{\partial q_j} = -\frac{\partial}{\partial q_j} \sum_{i=1}^{N} V_i = -\frac{\partial V}{\partial q_j}$$

$$\boldsymbol{\nabla} V \cdot d\boldsymbol{r} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right) V \cdot (dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k})$$

$$\boldsymbol{\nabla} V \cdot d\boldsymbol{r} = \left(\frac{\partial V}{\partial x}\hat{\imath} + \frac{\partial V}{\partial y}\hat{\jmath} + \frac{\partial V}{\partial z}\hat{k}\right) \cdot (dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k})$$

$$\boldsymbol{\nabla} V \cdot d\boldsymbol{r} = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = dV$$

For a function V(x,y,z)

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

Therefore, the Lagrange's Equation can be written as

$$Q_j - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} + \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j} - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} + \frac{\partial T}{\partial q_j} = 0$$

$$\Rightarrow \frac{d}{dt}\frac{\partial (T-V)}{\partial \dot{q}_j} - \frac{\partial (T-V)}{\partial q_j} = 0$$

$$\boxed{\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0}$$

Lagrange's Equation For conservative force

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

Where $L = T - V$ is called Lagrangian of the system.

This equation involves only Kinetic and potential energy which are scalar quantities.

Hence, we have developed our mechanics such that we do not require all information about the forces, which were necessary in Newtonian mechanics,

When we transform from space coordinates system to generalized coordinates system, the forces remain invariant.

Whereas the Lagrangian $L = T - V$ is invariant under coordinate transformation.

Lagrange's Equation For Velocity dependent potential

For a velocity dependent potential $U(\dot{q}_j, q_j)$ the generalize force can be derivable as

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_j}$$

Therefore, the Lagrange's Equation can be written as

$$Q_j - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \frac{\partial L}{\partial q_j} = 0$$

$$\Rightarrow \frac{d}{dt}\frac{\partial (T-U)}{\partial \dot{q}_j} - \frac{\partial (T-U)}{\partial q_j} = 0$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

Where $$L = T(\dot{q}_j, q_j) - U(\dot{q}_j, q_j)$$

The practical example of this case is motion of charged particle in electromagnetic field.

How to solve Problems using Lagrange's Equation

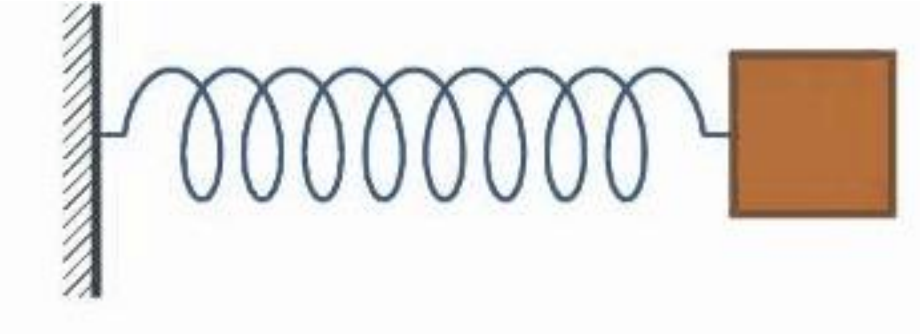
1. Identify the generalize coordinates (independent)
2. Express kinetic energy and Potential Energy into terms of independent coordinates
3. Find $L = T - V$
4. Use Lagrange's equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

q_j is independent coordinate

Lagrange's Mechanics Examples

A Particle of mass "*m*" attached to spring and pulled by a force "F" to a distance x. Find its equation of motion.



Solution: Kinetic energy = $T = \frac{1}{2} m\dot{x}^2$,

Potential energy = $V = \frac{1}{2} kx^2$

$\Rightarrow$ Lagrangian = $L = T - V = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$

Now Lagrange's Equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\left[\frac{1}{2} m\dot{x}^2 - \overset{0}{\cancel{\frac{1}{2} kx^2}}\right] - \frac{\partial}{\partial x}\left[\overset{0}{\cancel{\frac{1}{2} m\dot{x}^2}} - \frac{1}{2} kx^2\right] = 0$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \frac{d}{dt}(m\dot{x}) + kx = m\ddot{x} + kx = 0$$

$$\Rightarrow \; m\ddot{x} = -kx \quad \text{or} \quad m\ddot{\vec{x}} = -k\vec{x}$$

Simple Pendulum

For y coordinate

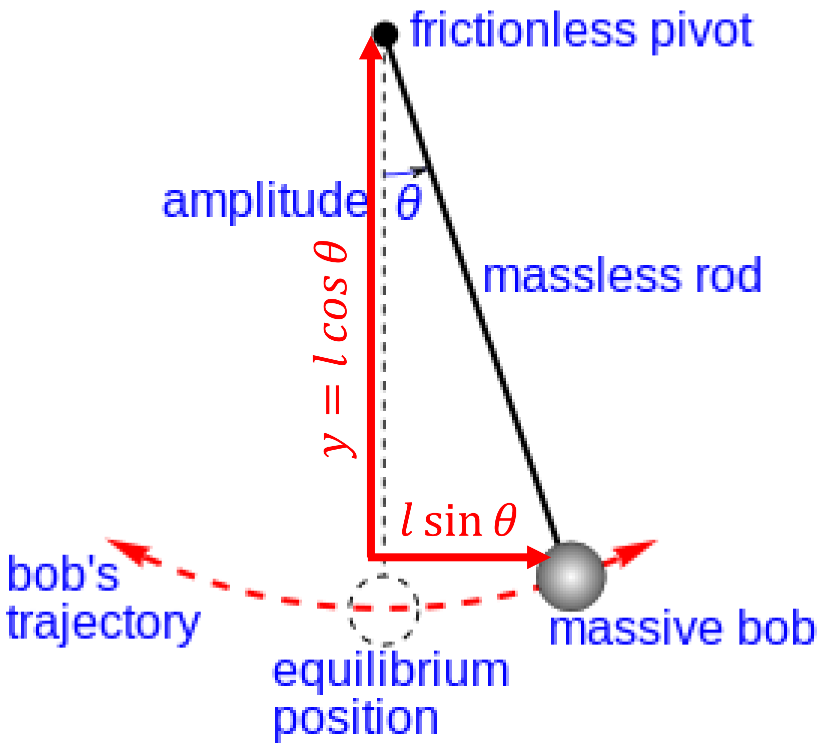
$$y = l\, cos\, \theta$$

$$x = l \sin\theta$$

$$T = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$V = -mgy = -mgl\cos\theta$$

Lagrangian $L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$



Independent coordinates is only θ

Therefore, Lagrange's equation can be written as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\frac{\partial}{\partial \dot{\theta}}\left[\frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta\right] - \frac{\partial}{\partial \theta}\left[\frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta\right] = 0$$

Simple Pendulum

$$\frac{d}{dt}\left[ml^2\dot{\theta}\right] - [-mgl\,sin\,\theta] = 0$$

$$ml^2\ddot{\theta} + mgl\,sin\,\theta = 0$$

$$ml^2\ddot{\theta} + mgl\,sin\,\theta = 0$$

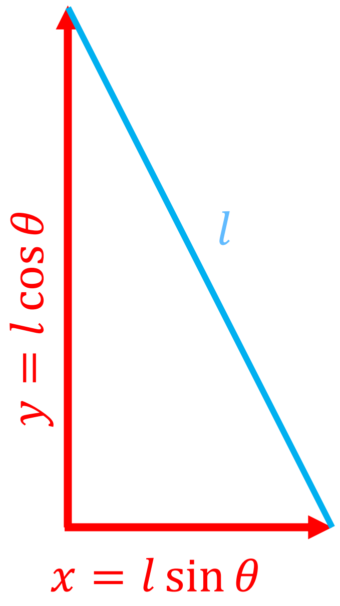
$$\boxed{\ddot{\theta} + \frac{g}{l}sin\,\theta = 0}$$

θ is very small $\quad sin\,\theta \approx \theta$

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

$$\ddot{\theta} = -\frac{g}{l}\theta$$

$$\boxed{\ddot{\theta} \propto -\theta}$$



The motion of simple pendulum will be simple hormonic motion.

Compound Pendulum

A rigid body capable to oscillate in a plane about a fix point is called compound pendulum.

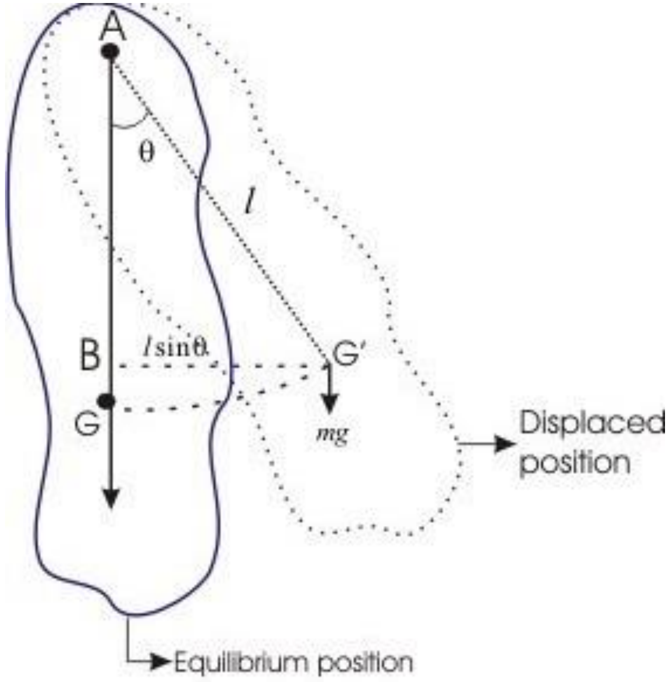
Let us consider a body of mass m suspended at pivot point "P". If "*l*" is the distance between suspension point and center of mass G. Where radius of gyrating is "*k*" (root mean square distance of particles or $k^2 = [r_1^2 + r_2^2 + r_3^2 + \cdots + r_n^2]/_{n})$ and $I = mk^2$

$$T = \frac{1}{2} I\dot{\theta}^2$$

$$V = -mgl\cos\theta$$

Lagrangian $\quad L = T - V = \frac{1}{2} I\dot{\theta}^2 + mgl\cos\theta$

Therefore, Lagrange's equation can be written as $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$



$$\mathrm{I}\ddot{\theta} + mgl\sin\theta = 0$$

$$\Rightarrow \ddot{\theta} + \frac{mgl}{\mathrm{I}}\sin\theta = 0 \text{ or } \Rightarrow \quad \ddot{\theta} + \frac{gl}{k^2}\sin\theta = 0$$

Projectile Motion

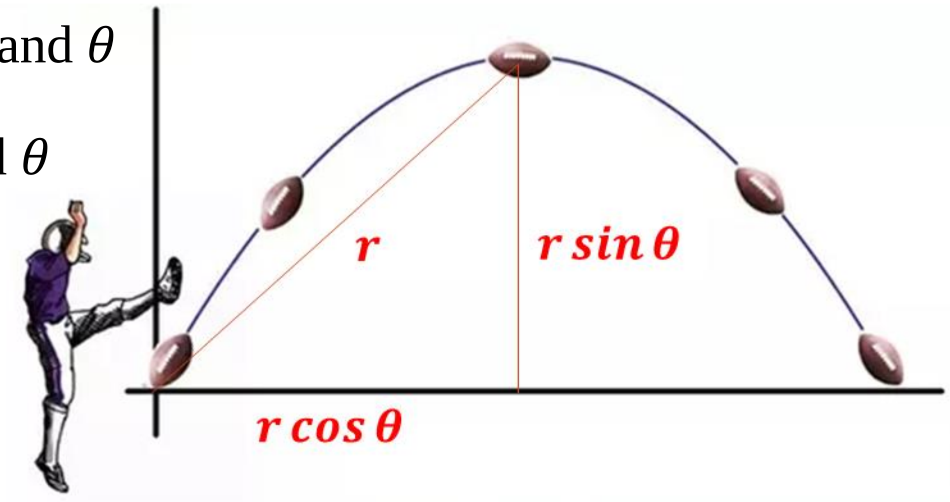
$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$V = mgy = mgr\sin\theta$$

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\sin\theta$$

In projectile motion we have tow generalized coordinates r and θ

Therefore, we must solve two Lagrange's equation for r and θ



$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$$

$$\boxed{\Rightarrow\ m\left(\ddot{r} - r\dot{\theta}^2\right) + mg\sin\theta = 0}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\boxed{\Rightarrow m\left(r\ddot{\theta} - 2\dot{r}\dot{\theta}\right) + mg\cos\theta = 0}$$

Lagrange's Equation for planetary motion

Lagrange's Equation of planetary motion under central potential V $= -m\mu/r$

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$V = -\frac{\mathrm{m\mu}}{\mathrm{r}}$$

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{m\mu}{r}$$

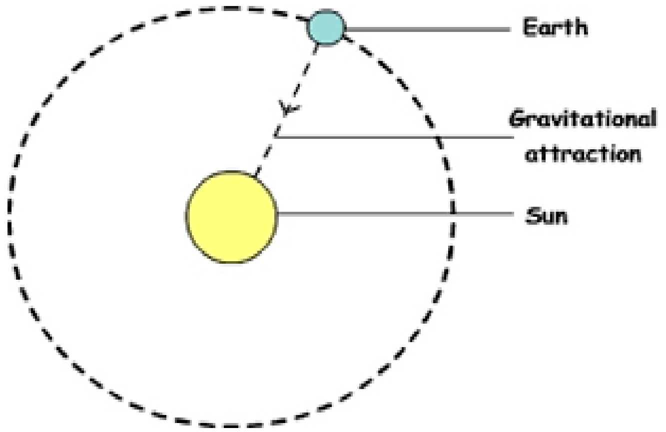
Generalized coordinates are "r" and "θ", Therefore we must solve two Lagrange's equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$$

$$\boxed{\Rightarrow\ m\left(\ddot{r} - r\dot{\theta}^2\right) + \frac{m\mu}{r^2} = 0}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\boxed{\Rightarrow \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0 \qquad \Rightarrow mr^2\dot{\theta} = L = constant}$$



Theorem

Prove that in a simple dynamical conservative system $T + V = \ constant$

Proof: for a conservative system

We must prove $\frac{d}{dt}[T + V] = 0$

We know that for conservative system $V = V(q_1, q_2, q_3, \dots \dots q_n)$

$T = T(q_1, q_2, q_3, \dots \dots q_n, \dot{q}_1, \dot{q}_2, \dot{q}_3 \dots \dots \dot{q}_n)$

Therefore $L = T - V = L(q_1, q_2, q_3, \dots \dots q_n, \dot{q}_1, \dot{q}_2, \dot{q}_3 \dots \dots \dot{q}_n)$

Now

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j} \sum_i^N \frac{1}{2} m_i \dot{r}_i^2$$

$$\frac{\partial L}{\partial \dot{q}_j} = \sum_i^N m_i \dot{r}_i \frac{\partial \dot{r}_i}{\partial \dot{q}_j}$$

$$\dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = \sum_i^N m_i \dot{r}_i \ \frac{\partial \dot{r}_i}{\partial \dot{q}_j} \dot{q}_j$$

Theorem

And $$\sum_j^n \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = \sum_{i,j} m_i \dot{r}_i \;\; \frac{\partial \dot{r}_i}{\partial \dot{q}_j} \dot{q}_j$$

$$\sum_j^n \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = \sum_i^N m_i \dot{r}_i \sum_j^n \frac{\partial \dot{r}_i}{\partial \dot{q}_j} \dot{q}_j$$

$$\sum_j^n \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = \sum_i^N m_i \dot{r}_i \sum_j^n \frac{\partial r_i}{\partial q_j} \dot{q}_j$$

because $\frac{\partial \dot{r}_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j}$ & $\dot{r}_i = \sum_j^n \frac{\partial r_i}{\partial q_j} \dot{q}_j$

$$\sum_j^n \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = \sum_i^N m_i \dot{r}_i^2 = 2 \sum_i^N \frac{1}{2} m_i \dot{r}_i^2$$

$$\boxed{\sum_j^n \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = 2T}$$

And $$\sum_j^n \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L = 2T - L$$

$$\sum_j^n \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L = 2T - T + V = T + V$$

Theorem

And $$\frac{d}{dt}\left[\sum_j^n \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L\right] = \sum_j^n \ddot{q}_j \frac{\partial L}{\partial \dot{q}_j} + \sum_j^n \dot{q}_j \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{d}{dt}L$$

$$\frac{d}{dt}\left[\sum_j^n \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L\right] = \sum_j^n \ddot{q}_j \frac{\partial L}{\partial \dot{q}_j} + \sum_j^n \dot{q}_j \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \sum_j^n\left[\frac{\partial L}{\partial q_j}\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j\right]$$

$$\frac{d}{dt}\left[\sum_j^n \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L\right] = \cancel{\sum_j^n \ddot{q}_j \frac{\partial L}{\partial \dot{q}_j}} + \sum_j^n \dot{q}_j \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \sum_j^n \frac{\partial L}{\partial q_j}\dot{q}_j - \cancel{\sum_j^n \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j}$$

$$\frac{d}{dt}\left[\sum_j^n \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L\right] = \sum_j^n \dot{q}_j \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \sum_j^n \frac{\partial L}{\partial q_j}\dot{q}_j$$

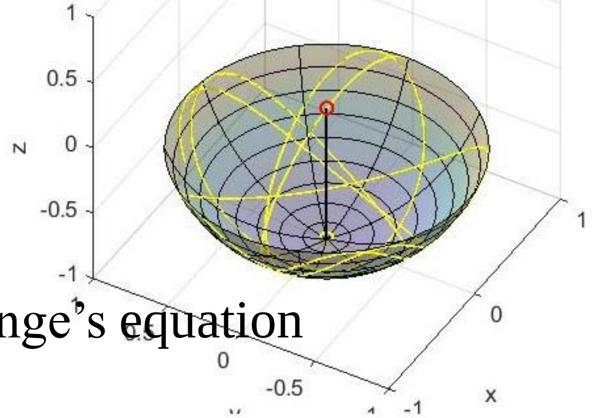
$$\frac{d}{dt}\left[\sum_j^n \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L\right] = \sum_j^n \left[\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j}\right]\dot{q}_j = 0$$

$$\left[\sum_j^n \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L\right] = constant$$

$$\left[\sum_j^n \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L\right] = 2T - L = T + V = constant$$

Spherical Pendulum

Solution: in the case of spherical pendulum the bob moves on a s smooth sphere of radius “r”. The position of the bob is located by spherical coordinates (r, θ, φ). The distance r of the bob from the center of the sphere on which the bod moves is radius (constant) of the sphere



$$T = \frac{1}{2} m r^2 (\dot{\theta}^2 + \sin^2\theta\, \dot{\varphi}^2) \quad \& \quad V = -mgr\cos\theta$$

$$L = T - V = \frac{1}{2} m r^2 \left(\dot{\theta}^2 + \sin^2\theta\ \dot{\varphi}^2\right) + mgr\cos\theta$$

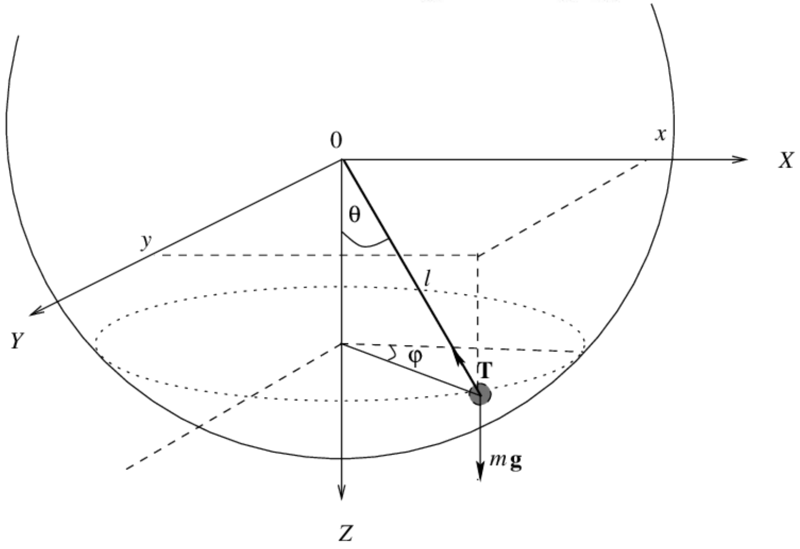
Generalized coordinates are θ and φ, Therefore we must solve two Lagrange’s equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\boxed{\Rightarrow\ m r^2 \left(\ddot{\theta} - \sin\theta\cos\theta\, \dot{\varphi}^2 + \frac{g}{r}\sin\theta\right) = 0}$$

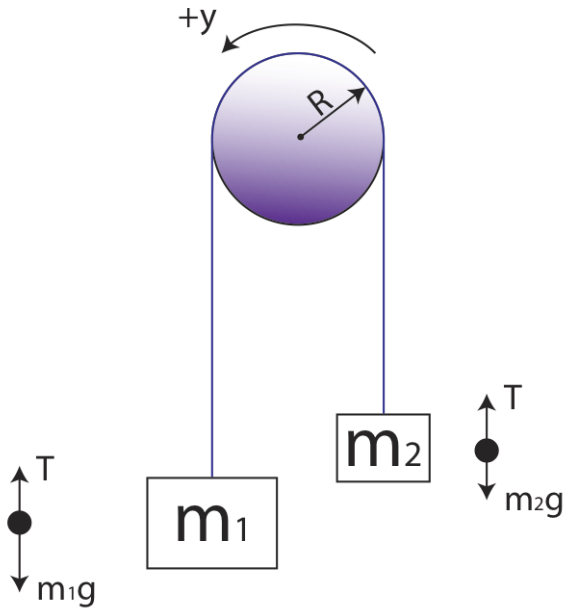
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0$$

$$\Rightarrow \frac{d}{dt}(m r^2\, sin^2\theta\, \dot{\varphi}) = 0 \qquad \boxed{\Rightarrow \ddot{\varphi} = -2\cot\theta\, \dot{\theta}\dot{\varphi}}$$



Atwood Machine

Atwood machine is a simple machine where two masses can move over a frictional less pully.



Equation of motion for m_1

$$T - m_1 g = -m_1 a\ldots\ldots\ldots\ldots\ldots\ldots\ldots..1$$

Equation of motion for m_2

$$T - m_2 g = m_2 a\ldots\ldots\ldots\ldots\ldots\ldots\ldots2$$

Subtraction Equation 1 from equation 2

$$T - m_2 g = m_2 a$$

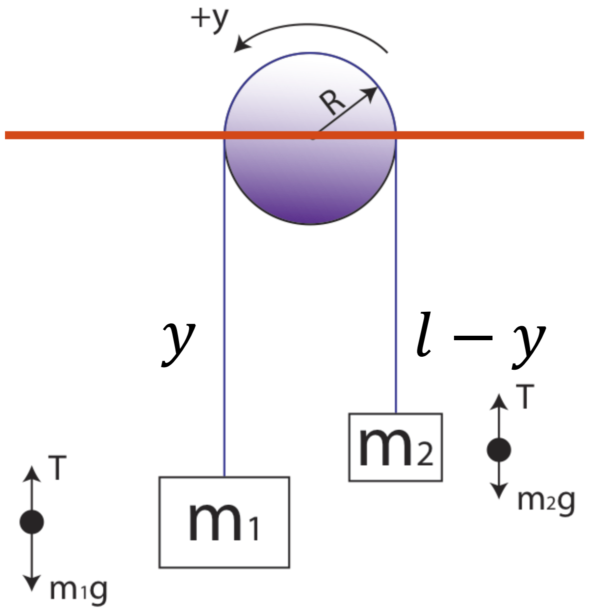
$$T - m_1 g = -m_1 a$$

$$-m_2 g + m_1 g = m_2 a + m_1 a$$

$$\boxed{\Rightarrow \mathrm{a} = \ddot{y} = g\frac{(m_1 - m_2)}{(m_1 + m_2)}}$$

Atwood Machine

Atwood machine is a simple machine where two masses can move over a frictional less pully.



$$T = \frac{1}{2}m_1\dot{y}^2 + \frac{1}{2}m_2\dot{y}^2 = \frac{1}{2}\dot{y}^2(m_1+m_2) \quad \&$$

$$V = -m_1gy - m_2g(l-y) = -gy(m_1-m_2) - m_2gl$$

$$\boxed{L = T - V = \frac{1}{2}\dot{y}^2(m_1+m_2) + gy(m_1-m_2) + m_2gl}$$

Generalized coordinate is q = y

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0$$

$$\Rightarrow \frac{d}{dt}[\dot{y}(m_1+m_2)] - g(m_1-m_2) = 0$$

$$\Rightarrow \ddot{y}(m_1+m_2) - g(m_1-m_2) = 0$$

$$\boxed{\Rightarrow \ddot{y} = g\frac{(m_1-m_2)}{(m_1+m_2)}}$$

Note: If we consider the motion of pully which is rotating about a fixed axis. The Kinetic energy must include a term

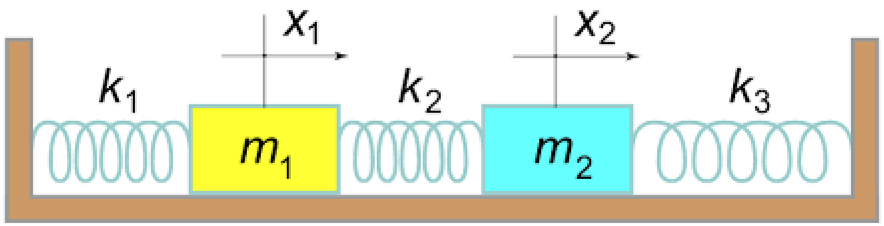
$$\frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}I\frac{\dot{y}^2}{R^2}$$

And $\ddot{y} = g\frac{(m_1-m_2)}{\left(m_1+m_2+\frac{I}{R^2}\right)}$

Two masses attached with springs

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_3x_2^2 + \frac{1}{2}k_2(x_1 - x_2)^2$$



$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \left[\frac{1}{2}k_1x_1^2 + \frac{1}{2}k_3x_2^2 + \frac{1}{2}k_2(x_1 - x_2)^2\right]$$

In this problem we have two degrees of freedom for x_1 and x_2

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = \frac{d}{dt}(m_1\dot{x}_1) - [-k_1x_1 - k_2(x_1 - x_2)]$$

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = m_1\ddot{x}_1 + x_1(k_1 + k_2) - k_2x_2}$$

And

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} - \frac{\partial L}{\partial x_2} = m_2\ddot{x}_2 + x_2(k_3 + k_2) - k_2x_1}$$

Derivative 8 Page No 30

If L is a Lagrangian for a system of n degree of freedom satisfying Lagrange's equation. Show by direct substitution that

$$\boxed{L' = L + \frac{d}{dt}F(q_1, q_2, q_3, \dots \dots q_n, t)}$$

also satisfies Lagrange's equating where F is any arbitrary but differentiable function of its arguments.

Solution: $$\frac{\partial L'}{\partial q_j} = \frac{\partial L}{\partial q_j} + \frac{\partial}{\partial q_j}\frac{d}{dt}F$$

And $$\frac{\partial L'}{\partial \dot{q}_j} = \frac{\partial L}{\partial \dot{q}_j} + \frac{\partial}{\partial \dot{q}_j}\frac{d}{dt}F$$

Since $$F(q_1, q_2, q_3, \dots \dots q_n, t)$$

$$\frac{dF}{dt} = \sum \frac{\partial F}{\partial q_j}\dot{q}_j + \frac{\partial F}{\partial t}$$

$$\frac{\partial}{\partial \dot{q}_j}\frac{dF}{dt} = \frac{\partial F}{\partial q_j}$$

Derivative 8 Page No 30

Therefore,
$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\frac{dF}{dt}\right) = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \frac{d}{dt}\frac{\partial F}{\partial q_j}$$

And
$$\boxed{\frac{\partial L'}{\partial q_j} = \frac{\partial L}{\partial q_j} + \frac{\partial}{\partial q_j}\frac{dF}{dt}}$$

Therefore,
$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_j} - \frac{\partial L'}{\partial q_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \frac{d}{dt}\frac{\partial F}{\partial q_j} - \frac{\partial L}{\partial q_j} - \frac{\partial}{\partial q_j}\frac{dF}{dt}$$

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_j} - \frac{\partial L'}{\partial q_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} + \left[\frac{d}{dt}\frac{\partial F}{\partial q_j} - \frac{d}{dt}\frac{\partial F}{\partial q_j}\right]$$

$$\boxed{\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_j} - \frac{\partial L'}{\partial q_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0}$$

Exercise: Problem 20

A particle of mass m moves in one dimension such that it has Lagrangian $L = \frac{m^2\dot{x}^4}{12} + m\dot{x}^2 V_x - V_x^2$

Where V is some differentiable function of x Find the equation of motion for x(t). Describe the physical nature of the system on the basis of this equation

$$\frac{\partial L}{\partial \dot{x}} = \frac{m^2\dot{x}^3}{3} + 2m\dot{x}V_x$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m^2\dot{x}^2\ddot{x} + 2m\ddot{x}V_x + 2m\dot{x}\frac{d}{dt}V_x = m^2\dot{x}^2\ddot{x} + 2m\ddot{x}V_x + 2m\dot{x}^2\frac{\partial V_x}{\partial x}$$

$$\frac{\partial L}{\partial x} = m\dot{x}^2\frac{\partial V_x}{\partial x} - 2V_x\frac{\partial V_x}{\partial x}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m^2\dot{x}^2\ddot{x} + 2m\ddot{x}V_x + 2m\dot{x}^2\frac{\partial V_x}{\partial x} - m\dot{x}^2\frac{\partial V_x}{\partial x} + 2V_x\frac{\partial V_x}{\partial x}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m^2\dot{x}^2\ddot{x} + 2m\ddot{x}V_x + m\dot{x}^2\frac{\partial V_x}{\partial x} + 2V_x\frac{\partial V_x}{\partial x}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m\dot{x}^2\left(m\ddot{x} + \frac{\partial V_x}{\partial x}\right) + 2V_x\left(m\ddot{x} + \frac{\partial V_x}{\partial x}\right) = \left(m\ddot{x} + \frac{\partial V_x}{\partial x}\right)(m\dot{x}^2 + 2V_x) = 0$$

$$m\ddot{x} = -\frac{\partial V_x}{\partial x} \qquad \& \qquad \frac{1}{2}m\dot{x}^2 + V_x = 0$$

Exercise: Problem 20

Notice that $$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + V_x\right) = m\dot{x}\ddot{x} + \frac{\partial V}{\partial x}\dot{x} = \dot{x}\left(m\ddot{x} + \frac{\partial V_x}{\partial x}\right)$$

So, if we denote

$$E = \frac{1}{2}m\dot{x}^2 + V_x$$

Equation become $$\dot{x}\left(m\ddot{x} + \frac{\partial V_x}{\partial x}\right)(m\dot{x}^2 + 2V_x) = \dot{E}(2E) = 0$$

Or $$\dot{E}E = 0$$

If we forget about the trivial case where the particles is not moving $\dot{x} \neq 0$

This becomes

$$\dot{E}E = 0$$

Notice that $$\frac{d}{dt}E^2 = 2E\dot{E} = 0$$

So, this is the motion where the quantity E^2 is conserved.

Exercise: Problem 20

In another nontrivial case where $E \neq 0$ then we can get $\dot{E} = 0$

Since we have shown that

$$\dot{\mathrm{E}} = \dot{\mathrm{x}}\left(\mathrm{m}\ddot{\mathrm{x}} + \frac{\partial V_x}{\partial x}\right)$$

The Equation of motion in this case is

$$m\ddot{x} + \frac{\partial V_x}{\partial x} = 0$$

This is motion in conservative field force described by potential V_x

In the case $E \neq 0$ we have

Which mean that E is some given constant, In the case of E=0 again E is a constant just in this particular case that constant is equation to zero.

Thus, we can use that in both cases the equation if motion is given by

$$\frac{1}{2}m\dot{x}^2 + V_x = E$$

Exercise: Problem 20

Where E=constant expressing $\dot{x}$ we get

$$\dot{x} = \pm\sqrt{\frac{2(E - V_x)}{m}}$$

Integrating this equation, we can obtain x(t)

This is the motion of the particle in one dimension in the conservative potential V. X(t) is obtained from

$$\frac{1}{2}m\dot{x}^2 + V_x = E$$

Knowing what V_x is.

Chapter 2

Lecture 3

Lagrange's Mechanics

Dr. Akhlaq Hussain

Velocity dependent Potential

Charge Particle in Electromagnetic field

Solution: First we will find the potential using maxwells equations.

$\bar{\nabla} \times \bar{E} + \frac{\partial \bar{B}}{\partial t} = 0$ Maxwell –Faraday Law of electromagnetic induction.

$$\Rightarrow \bar{\nabla} \times \bar{E} + \frac{\partial}{\partial t}(\bar{\nabla} \times \bar{A}) = 0$$

$$\Rightarrow \bar{\nabla} \times \left[\bar{E} + \frac{\partial \bar{A}}{\partial t}\right] = 0$$

And $\bar{\nabla} \times \bar{\nabla}\varphi = 0$ For any scalar potential

$$\Rightarrow \bar{E} + \frac{\partial \bar{A}}{\partial t} = -\bar{\nabla}\varphi$$

$$\Rightarrow \bar{E} = -\bar{\nabla}\varphi - \frac{\partial \bar{A}}{\partial t}$$

$\varphi \rightarrow scalar\ Potential$ & $\bar{A} \rightarrow vector\ potential$

Velocity dependent Potential

$$\bar{F} = q(\bar{E} + \bar{v} \times \bar{B}) = q[-\bar{\nabla}\varphi - \frac{\partial \bar{A}}{\partial t} + \bar{v} \times (\bar{\nabla} \times \bar{A})]$$

For 1-dimensional

$$\bar{F}_x = q[-\frac{\partial \varphi}{\partial x} - \frac{\partial A_x}{\partial t} + \{\bar{v} \times (\bar{\nabla} \times \bar{A})\}_x]$$

To find the third Term

$$\left|\bar{v} \times \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}\right|_x = \left|\bar{v} \times [\hat{\imath}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})\right|_x$$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ v_x & v_y & v_z \\ \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} & \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} & \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{vmatrix}_x = v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x})$$

Solving for x direction only

Velocity dependent Potential

$$\Rightarrow \{\bar{v} \times (\bar{\nabla} \times \bar{A})\}_x = v_y \frac{\partial A_y}{\partial x} - v_y \frac{\partial A_x}{\partial y} + v_z \frac{\partial A_z}{\partial x} - v_z \frac{\partial A_x}{\partial z}$$

$$\Rightarrow \{\bar{v} \times (\bar{\nabla} \times \bar{A})\}_x = v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_y}{\partial x} + v_z \frac{\partial A_z}{\partial x} - v_x \frac{\partial A_x}{\partial x} - v_y \frac{\partial A_x}{\partial y} - v_z \frac{\partial A_x}{\partial z}$$

$$\Rightarrow \{\bar{v} \times (\bar{\nabla} \times \bar{A})\}_x = \frac{\partial}{\partial x}(\bar{v} \cdot \bar{A}) - [\frac{\partial A_x}{\partial x}\frac{dx}{dt} + \frac{\partial A_x}{\partial y}\frac{dy}{dt} + \frac{\partial A_x}{\partial z}\frac{dz}{dt}]$$

$$\Rightarrow \{\bar{v} \times (\bar{\nabla} \times \bar{A})\}_x = \frac{\partial}{\partial x}(\bar{v} \cdot \bar{A}) - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t}$$

$$\Rightarrow \bar{F}_x = q[-\frac{\partial \varphi}{\partial x} - \frac{\partial A_x}{\partial t} + \frac{\partial}{\partial x}(\bar{v} \cdot \bar{A}) - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t}]$$

$$\Rightarrow \bar{F}_x = q[-\frac{\partial \varphi}{\partial x} + \frac{\partial}{\partial x}(\bar{v} \cdot \bar{A}) - \frac{dA_x}{dt}]$$

$$\Rightarrow \bar{F}_x = q[-\frac{\partial}{\partial x}\{\varphi - (\bar{v} \cdot \bar{A})\} - \frac{dA_x}{dt}]$$

$$\bar{F} = q[-\bar{\nabla}\{\varphi - (\bar{v} \cdot \bar{A})\} - \frac{d}{dt}\frac{\partial}{\partial v}(\bar{v} \cdot \bar{A})]$$

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\frac{dx}{dt} + \frac{\partial A_x}{\partial y}\frac{dy}{dt} + \frac{\partial A_z}{\partial x}\frac{dz}{dt}$$

$$\Rightarrow -\left[\frac{\partial A_x}{\partial x}\frac{dx}{dt} + \frac{\partial A_x}{\partial y}\frac{dy}{dt} + \frac{\partial A_z}{\partial x}\frac{dz}{dt}\right] = -\frac{dA_x}{dt} + \frac{\partial A_x}{\partial t}$$

And

$$\{\bar{v} \times (\bar{\nabla} \times \bar{A})\}_y = \frac{\partial}{\partial y}(\bar{v} \cdot \bar{A}) - \frac{dA_y}{dt} + \frac{\partial A_y}{\partial t}$$

$$\{\bar{v} \times (\bar{\nabla} \times \bar{A})\}_z = \frac{\partial}{\partial z}(\bar{v} \cdot \bar{A}) - \frac{dA_z}{dt} + \frac{\partial A_z}{\partial t}$$

Velocity dependent Potential

$$\Rightarrow \bar{F} = q\left[-\bar{\nabla}\{\varphi - (\bar{A}\cdot\bar{v})\} + \frac{d}{dt}\frac{\partial}{\partial v}\{\varphi - (\bar{A}\cdot\bar{v})\}\right]$$

$$\Rightarrow \bar{F} = -\bar{\nabla}U + \frac{d}{dt}\frac{\partial}{\partial v}U$$

Therefore,

$$\mathrm{U} = q\varphi - q(\bar{A}\cdot\bar{v})$$

$$\frac{\partial\varphi}{\partial v} = 0$$

And

$$\frac{\partial}{\partial v}(\bar{A}\cdot\bar{v}) = \bar{A}\cdot\frac{\partial}{\partial v}\bar{v} = \bar{A}$$

Solve Lagrange's equation of Charge particle in electromagnetic field

Kinetic energy $T = \frac{1}{2}m\dot{r}^2$

Potential Energy $\mathrm{U} = q\varphi - q(\bar{A}\cdot\bar{v}) = q\varphi - q(\bar{A}\cdot\dot{\bar{r}})$

Lagrangian

$$L = T - U = \frac{1}{2}m\dot{r}^2 - q\varphi + q(\bar{A}\cdot\dot{\bar{r}})$$

Using Lagrange's equation $\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} + q\bar{A}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r} + q\frac{d\bar{A}}{dt}$$

Now $\frac{\partial L}{\partial r} = -q\frac{\partial \varphi}{\partial r} + q\frac{\partial}{\partial r}(\bar{A}\cdot\dot{\bar{r}}) = -q\bar{\nabla}\varphi + q\bar{\nabla}(\bar{A}\cdot\dot{\bar{r}})$

Solve Lagrange's equation of Charge particle in electromagnetic field

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = m\ddot{r} + q\frac{d\bar{A}}{dt} + q\bar{\nabla}\varphi - q\bar{\nabla}(\bar{A}\cdot\dot{\bar{r}}) = 0$$

$$\Rightarrow m\ddot{r} + q[\frac{\partial A}{\partial t} + (\dot{\bar{r}}\cdot\bar{\nabla})\bar{A}] + q\bar{\nabla}\varphi - q\bar{\nabla}(\bar{A}\cdot\dot{\bar{r}}) = 0$$

$$\Rightarrow m\ddot{r} - q\left(-\bar{\nabla}\varphi - \frac{\partial A}{\partial t}\right) - q[\bar{\nabla}(\dot{\bar{r}}\cdot\bar{A}) - (\dot{\bar{r}}\cdot\bar{\nabla})\bar{A}] = 0$$

$$\Rightarrow m\ddot{r} - q\bar{E} - q[\dot{\bar{r}}\times(\bar{\nabla}\times\bar{A})] = 0$$

$$\Rightarrow m\ddot{r} - q\bar{E} - q[\bar{v}\times\bar{B}] = 0$$

$$\Rightarrow m\ddot{r} = q\bar{E} + q[\bar{v}\times\bar{B}]$$

$$\frac{d\bar{A}}{dt} = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial r}\frac{dr}{dt}$$

$$\frac{d\bar{A}}{dt} = \frac{\partial A}{\partial t} + \frac{dr}{dt}\frac{\partial A}{\partial r}$$

$$\frac{d\bar{A}}{dt} == \frac{\partial A}{\partial t} + \dot{r}\frac{\partial A}{\partial r}$$

$$\frac{d\bar{A}}{dt} = \frac{\partial A}{\partial t} + \dot{r}\,.\,\bar{\nabla}\bar{A}$$

$$\frac{d\bar{A}}{dt} = \frac{\partial A}{\partial t} + (\dot{r}\,.\,\bar{\nabla})\bar{A}$$

Double Pendulum

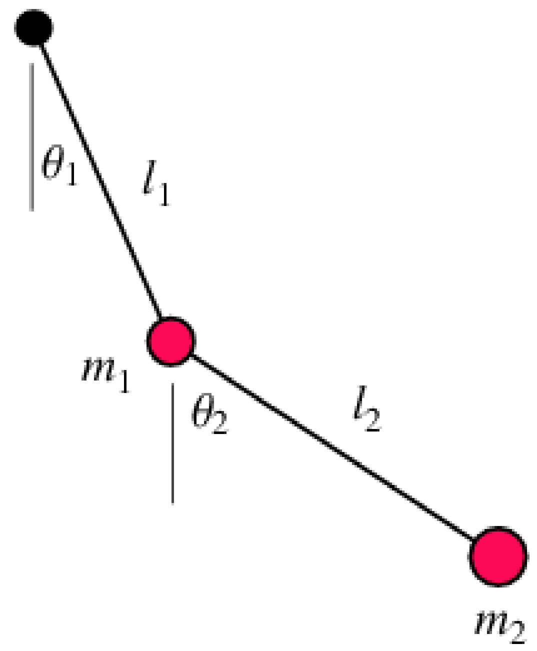
Consider a double pendulum with masses m_1 and m_2 attached by rigid mass less wires of length l_1 and l_2 the angle they made with vertical axis are θ and φ as illustrated in figure. Position of Bobs

$x_1 = l_1 \sin\theta$ & $y_1 = l_1 \cos\theta$

$x_2 = l_1 \sin\theta + l_2 \sin\varphi$ & $y_2 = l_1 \cos\theta + l_2 \cos\varphi$

$\dot{x}_1 = l_1\dot{\theta}\, cos\,\theta$ & $\dot{y}_1 = -l_1\dot{\theta} \sin\theta$

$\dot{x}_2 = l_1\dot{\theta}\, cos\,\theta + l_2\dot{\varphi}\, cos\,\varphi$ & $\dot{y}_2 = -(l_1\dot{\theta}\, sin\,\theta + l_2\dot{\varphi}\, sin\,\varphi\ )$



Kinetic energy

$$T = \frac{1}{2}m_1 l_1^2\dot{\theta}^2 + \frac{1}{2}m_2 l_1^2\dot{\theta}^2 + \frac{1}{2}m_2 l_2^2\dot{\varphi}^2 + m_2 l_1 l_2 \dot{\theta}\dot{\varphi}\cos(\theta - \varphi)$$

Potential Energy=

$$V = -m_1 g y_1 - m_2 g y_2 = -m_1 g l_1 \cos\theta - m_2(l_1\cos\theta + l_2\cos\varphi)$$

Double Pendulum

$$L = \frac{1}{2} m_1 l_1^2 \dot{\theta}^2 + \frac{1}{2} m_2 l_1^2 \dot{\theta}^2 + \frac{1}{2} m_2 l_2^2 \dot{\varphi}^2 + m_2 l_1 l_2 \dot{\theta} \dot{\varphi} \cos(\theta - \varphi) + m_1 g l_1 \cos\theta + m_2 g (l_1 \cos\theta + l_2 \cos\varphi)$$

In this system we have tow degrees of freedom (θ, φ) Therefore to solve this system we need to solve two sperate Lagrange's equations.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \qquad \& \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = m_1 l_1^2 \dot{\theta} + m_2 l_1^2 \dot{\theta} + m_2 l_1 l_2 \dot{\varphi} \cos(\theta - \varphi)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = (m_1 + m_2) l_1^2 \ddot{\theta} + m_2 l_1 l_2 \ddot{\varphi} \cos(\theta - \varphi) - m_2 l_1 l_2 \dot{\varphi} \left(\dot{\theta} - \dot{\varphi}\right) \sin(\theta - \varphi)$$

$$\frac{\partial L}{\partial \theta} = -m_2 l_1 l_2 \dot{\theta} \dot{\varphi} \sin(\theta - \varphi) - (m_1 + m_2) g l_1 \sin\theta$$

Double Pendulum

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow (m_1 + m_2)l_1\ddot{\theta} + m_2 l_2 \ddot{\varphi}\cos(\theta - \varphi) - m_2 l_2 \dot{\varphi}^2 \sin(\theta - \varphi) + (m_1 + m_2)g\sin(\theta - \varphi) = 0$$

For $\varphi = \dot{\varphi} = \ddot{\varphi} = 0$

$$\Rightarrow (m_1 + m_2)l_1\ddot{\theta} + (m_1 + m_2)g\sin\theta = 0$$

$$\Rightarrow l_1\ddot{\theta} + g\sin\theta = 0$$

Double Pendulum

For $$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0$$

$$\frac{\partial L}{\partial \dot{\varphi}} = m_2 l_2^2 \dot{\varphi} + m_2 l_1 l_2 \dot{\theta}\cos(\theta - \varphi)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = m_2 l_2^2 \ddot{\varphi} + m_2 l_1 l_2 \ddot{\theta}\cos(\theta - \varphi) - m_2 l_1 l_2 \dot{\theta}\left(\dot{\theta} - \dot{\varphi}\right)\sin(\theta - \varphi)$$

$$\frac{\partial L}{\partial \theta} = m_2 l_1 l_2 \dot{\theta}\dot{\varphi}\sin(\theta - \varphi) - m_2 g l_2 \sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0$$

$$\Rightarrow m_2 l_2 \ddot{\varphi} + m_2 l_1 \ddot{\theta}\cos(\theta - \varphi) - m_2 l_1 \dot{\theta}^2 \sin(\theta - \varphi) + m_2 g \sin\varphi = 0$$

Which is equation of motion for m_2

For = θ $= \dot{\theta} = \ddot{\theta} = 0$

$\Rightarrow m_2 l_2 \ddot{\varphi} + m_2 g \sin\varphi = 0$ or $\ddot{\varphi} + \frac{g}{l_2}\sin\varphi = 0$

Nielsen Form Of Lagrange's Equation

Show that Lagrange's Equation $Q_j = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}$ can also be written as

$Q_j = \frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j}$ known as Nielsen Form of Lagrange's equations.

Solution: Since $T = T(q_i, \dot{q}_i)$

$$\frac{dT}{dt} = \sum_i \left[\frac{\partial T}{\partial q_i}\dot{q}_i + \frac{\partial T}{\partial \dot{q}_i}\ddot{q}_i\right]$$

$$\Rightarrow \dot{T} = \sum_i \left[\frac{\partial T}{\partial q_i}\dot{q}_i + \frac{\partial T}{\partial \dot{q}_i}\ddot{q}_i\right]$$

$$\Rightarrow \frac{\partial \dot{T}}{\partial \dot{q}_j} = \sum_i \frac{\partial}{\partial \dot{q}_j}\frac{\partial T}{\partial q_i}\dot{q}_i + \frac{\partial T}{\partial q_j} + \sum_i \frac{\partial}{\partial \dot{q}_j}\frac{\partial T}{\partial \dot{q}_i}\ddot{q}_i$$

$$\Rightarrow \frac{\partial \dot{T}}{\partial \dot{q}_j} = \sum_i \left[\frac{\partial}{\partial q_i}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\dot{q}_i + \frac{\partial}{\partial \dot{q}_i}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\ddot{q}_i\right] + \frac{\partial T}{\partial q_j} = \sum_i \left[\frac{\partial}{\partial q_i}X\dot{q}_i + \frac{\partial}{\partial \dot{q}_i}X\ddot{q}_i\right] + \frac{\partial T}{\partial q_j}$$

$$\Rightarrow \frac{\partial \dot{T}}{\partial \dot{q}_j} = \frac{dX}{dt} + \frac{\partial T}{\partial q_j} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} + \frac{\partial T}{\partial q_j} \qquad \text{for } X = X(q_i, \dot{q}_i)$$

Nielsen Form Of Lagrange's Equation

$$\Rightarrow \frac{\partial \dot{T}}{\partial \dot{q}_j} = \frac{dX}{dt} + \frac{\partial T}{\partial q_j} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} + \frac{\partial T}{\partial q_j} \qquad \text{for } X = X(q_i, \dot{q}_i)$$

Now

$$\Rightarrow \frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} + \frac{\partial T}{\partial q_j} - 2\frac{\partial T}{\partial q_j}$$

$$\Rightarrow \frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j$$

$$\Rightarrow \frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j} = Q_j$$

INVARIANT UNDER POINT TRANSFORMATIONS

Let $q_1, q_2, q_3, \ldots \ldots q_n$ be a set of independent generalized coordinates for a system of n degrees of freedom with a Lagrangian $L(q, \dot{q}, t)$. Suppose we transform to another set of independent coordinates $s_1, s_2, s_3, \ldots \ldots s_n$ by mean of transformation equations.

$$q_i = q_i(s_1, s_2, s_3, \ldots \ldots s_n, t)$$

(such a transformation is called a point transformation.) show that if the Lagrangian function is expressed as a function of $s_1, s_2, s_3, \ldots \ldots s_n$ and t through the equations of transformation, then L satisfies Lagrange's equation with respect to the s coordinates

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{s}_j} - \frac{\partial L}{\partial s_j} = 0$$

In other words, the form of Lagrange's equitation is invariant under point transformations.

INVARIANT UNDER POINT TRANSFORMATIONS

$$q_i = q_i(s_1, s_2, s_3, \dots \dots s_n, t)$$

$$\dot{q}_i = \sum_j \frac{\partial q_i}{\partial s_j}\dot{s}_j + \frac{\partial q_i}{\partial t}$$

And $L = L(q_i, \dot{q}_i, t) = L\left\{q_i(s_j, t), \sum_j \frac{\partial q_i}{\partial s_j}\dot{s}_j + \frac{\partial q_i}{\partial t}, t\right\}$

$$\frac{\partial L}{\partial s_j} = \sum_i \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial s_j} + \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial}{\partial s_j}\left(\sum_j \frac{\partial q_i}{\partial s_j}\dot{s}_j + \frac{\partial q_i}{\partial t}\right) + \frac{\partial L}{\partial t}\cancelto{0}{\frac{\partial t}{\partial s_j}}$$

$$\frac{\partial L}{\partial s_j} = \sum_i \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial s_j} + \sum_{i,j} \frac{\partial L}{\partial \dot{q}_i}\frac{\partial^2 q_i}{\partial {s_j}^2}\dot{s}_j + \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial^2 q_i}{\partial s_j \partial t}$$

Now $\dfrac{\partial L}{\partial \dot{s}_k} = \sum_i \dfrac{\partial L}{\partial q_i}\dfrac{\partial q_i}{\dot{s}_k} + \sum_i \dfrac{\partial L}{\partial \dot{q}_i}\dfrac{\partial \dot{q}_i}{\partial \dot{s}_k} + \dfrac{\partial L}{\partial t}\dfrac{\partial t}{\partial \dot{s}_k}$

$$\frac{\partial L}{\partial \dot{s}_k} = \sum_i \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\dot{s}_k} + \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial}{\partial \dot{s}_k}\left(\sum_j \frac{\partial q_i}{\partial s_j}\dot{s}_j + \frac{\partial q_i}{\partial t}\right) + \frac{\partial L}{\partial t}\frac{\partial t}{\partial \dot{s}_k}$$

$$\frac{\partial L}{\partial \dot{s}_k} = \sum_{i,j} \frac{\partial L}{\partial \dot{q}_i}\frac{\partial q_i}{\partial s_j}\frac{\partial \dot{s}_j}{\partial \dot{s}_k}$$

INVARIANT UNDER POINT TRANSFORMATIONS

$$\frac{\partial L}{\partial \dot{s}_k} = \sum_{i,j} \frac{\partial L}{\partial \dot{q}_i} \frac{\partial q_i}{\partial s_j} \delta_{jk} \qquad where\, \delta_{jk} \begin{cases} 1\; for\; j = k \\ 0\; for\; j \neq k \end{cases}$$

$$\frac{\partial L}{\partial \dot{s}_k} = \sum_i \frac{\partial L}{\partial \dot{q}_i} \frac{\partial q_i}{\partial s_k} \delta_{kk} = \sum_i \frac{\partial L}{\partial \dot{q}_i} \frac{\partial q_i}{\partial s_k}$$

And for $\dot{s}_j$

$$\frac{\partial L}{\partial \dot{s}_j} = \sum_i \frac{\partial L}{\partial \dot{q}_i} \frac{\partial q_i}{\partial s_j}$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{s}_j} = \sum_i \frac{d}{dt} \left[\frac{\partial L}{\partial \dot{q}_i} \frac{\partial q_i}{\partial s_j} \right]$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{s}_j} = \sum_i \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \frac{\partial q_i}{\partial s_j} + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \left(\frac{\partial q_i}{\partial s_j} \right) \right]$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{s}_j} = \sum_i \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \frac{\partial q_i}{\partial s_j} + \frac{\partial L}{\partial \dot{q}_i} \frac{\partial}{\partial s_j} \left(\frac{d q_i}{dt} \right) \right]$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{s}_j} = \sum_i \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \frac{\partial q_i}{\partial s_j} + \frac{\partial L}{\partial \dot{q}_i} \frac{\partial}{\partial s_j} \left(\sum_j \frac{\partial q_i}{\partial s_j} \dot{s}_j + \frac{\partial q_i}{\partial t} \right) \right]$$

INVARIANT UNDER POINT TRANSFORMATIONS

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}_j} = \sum_i \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial q_i}{\partial s_j} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial}{\partial s_j}\left(\sum_j \frac{\partial q_i}{\partial s_j}\dot{s}_j + \frac{\partial q_i}{\partial t}\right)\right]$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}_j} = \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial q_i}{\partial s_j} + \sum_{i,j}\frac{\partial L}{\partial \dot{q}_i}\frac{\partial^2 q_i}{{\partial s_j}^2}\dot{s}_j + \sum_{i,j}\frac{\partial L}{\partial \dot{q}_i}\cancel{\frac{\partial^2 q_i}{\partial s_j}}\frac{\partial \dot{s}_j}{\partial s_j} + \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial^2 q_i}{\partial s_j \partial t}$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{s}_j} - \frac{\partial L}{\partial s_j} = \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial q_i}{\partial s_j} + \sum_{i,j}\frac{\partial L}{\partial \dot{q}_i}\cancel{\frac{\partial^2 q_i}{{\partial s_j}^2}}\dot{s}_j + \sum_i \frac{\partial L}{\partial \dot{q}_i}\cancel{\frac{\partial^2 q_i}{\partial s_j \partial t}} - \sum_i \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial s_j} - \sum_{i,j}\cancel{\frac{\partial L}{\partial \dot{q}_i}}\frac{\partial^2 q_i}{{\partial s_j}^2}\dot{s}_j - \sum_i \frac{\partial L}{\partial \dot{q}_i}\cancel{\frac{\partial^2 q_i}{\partial s_j \partial t}}$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{s}_j} - \frac{\partial L}{\partial s_j} = \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial q_i}{\partial s_j} - \sum_i \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial s_j}$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{s}_j} - \frac{\partial L}{\partial s_j} = \sum_i \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}\right]\frac{\partial q_i}{\partial s_j} = \sum_i [0]\frac{\partial q_i}{\partial s_j} = 0$$

Hence the Lagrange’s equitation is invariant under point transformations.

Rayleigh's Dissipation Force

For a system where force is derivable from a scalar potential Lagrange’s eq is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

In case if force is not derivable from scalar potential or if the force has a component, which is not derivable from potential such as $F \neq -\nabla V$

Then $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j$$ where is not derivable from scaler potential.

Such a situation often arises when frictional forces are present. Since the frictional force is proportional to the velocity

$$F_{f_x} = -k_x v_x$$

Rayleigh's Dissipation Force

The frictional force of this type may be derivable a function $\mathcal{F}$ known as Raleigh dissipation function

$$\mathcal{F}=\frac{1}{2}kv^2=\frac{1}{2}\sum_i\left(k_x v_{ix}^2+k_y v_{iy}^2+k_z v_{iz}^2\right)$$

$$F_{f_x}=-\frac{\partial\mathcal{F}}{\partial v_x}=-\nabla_v\mathcal{F}$$

Now the work done by such force is

$$dw_f=-F_f.dr=-F_f.vdt=kv^2dt$$

Will be the amount of energy dissipated due to friction.

The component of the generalized force resulting from the force of friction

$$Q_j=\sum_i F_i\frac{\partial r_i}{\partial q_j}=-\sum_i\frac{\partial\mathcal{F}}{\partial v}\frac{\partial r_i}{\partial q_j}=-\sum_i\frac{\partial\mathcal{F}}{\partial\dot{r}_i}\frac{\partial\dot{r}_i}{\partial\dot{q}_j}$$

Rayleigh's Dissipation Force

$$Q_j = -\sum_i \frac{\partial \mathcal{F}}{\partial \dot{r}_i} \frac{\partial \dot{r}_i}{\partial \dot{q}_j}$$

$$Q_j = -\frac{\partial \mathcal{F}}{\partial \dot{q}_j} \qquad \text{where} \qquad \mathcal{F} = \mathcal{F}(\dot{r}_i)$$

An example is Stoke's law whereby a sphere of radius "a" moving at a speed "*V*" in a medium of viscosity η experience the friction drag force. The Lagrange's equation for dissipative system

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = 0$$

For such system two scalar functions "L" and "$\mathcal{F}$" should be defined to find the equation of motion.

Solve

9. The electromagnetic field is invariant under a gauge transformation of the scalar vector potential given by

$$A \to A + \nabla\psi(r,t) \qquad \& \qquad \varphi \to \varphi - \frac{1}{c}\frac{\partial\varphi}{\partial t}$$

Where ψ is arbitrary(but differentiable). What effect does this gauge transformation have on the Lagrangian of a particle moving in the electromagnetic field? Is the motion affected?

15. A point particle moves in space under the influence of a force derivable from a generalized potential of the form

$$U(r,v) = V(r) + \sigma.L$$

where r is the radius vector from a fixed point, L is the angular momentum about that point and σ is a fixed vector in space

a) Find the component of force on the particle in both cartesian and spherical polar coordinates on the basis of Eq. 1.58

b) Show that the component in the two coordinate systems are related to each other as in eq. 1.49.

c) Obtain the equation of motion in spherical polar coordinates.

16. A particle moves in a plane under the influence of a force action towards a centre of force, whose magnitude is

$$F = \frac{1}{r^2}\left(1 - \frac{\dot{r}^2 - 2\ddot{r}r}{c^2}\right)$$

Where r is the distance of particle to the centre of force.. Find the generalized potential that will result in such a force and form that the Lagrangian for the motion in a plane. (The expression for F represents the force between two charges in Weber's electrodynamics.)

Instructions

- Writing Style/font "Time New Roman", font size 12
- Students are advised to submit two files a word and Pdf copy of the assignment.
- Deadline of Submission is 4PM., Friday 25th December.
- Send your assignment on email address akhlaq.hussain@uop.edu.pk .
- Use the email addresses for submission which you provided previously.
- Files from any other email address will not be acceptable.
- If the author of the file is same or files are created through one computer or Cell phone, both assignments will be rejected.
- File name Must be "Roll Number-Name" e.g. 01-Akhlaq Hussain
- Strictly follow the instruction otherwise your assignment will not be considered.
- The title Page (Page 1) must be as following.

Assignment: **Classical Mechanics**
Roll No:________________________________
Name: ________________________________
E-Mail Address: ____________________________
Batch Number: ____________________________